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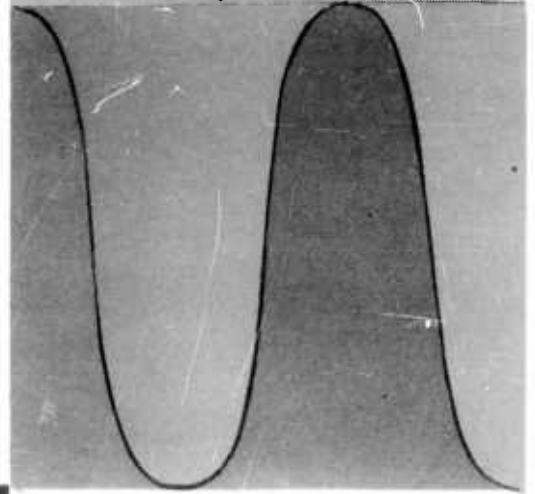
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ON THE SUPPLY OF SPARES TO INSURE
ADEQUATE TESTING

Sam C. Saunders

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ON THE SUPPLY OF SPARES TO INSURE ADEQUATE TESTING

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0. Introduction.

In this note we consider the problem of providing the minimum number of spares for one equipment which is to be tested for a given number of hours such that we will suffer a sufficient number of breakdowns or replacements to exhaust our stock of spares with probability not greater than a specified level. Note that unless a guarantee period (i. e. an initial period of life in which breakdown cannot occur) is known it is not ever possible to provide enough spares to insure i. e. with probability zero that breakdowns and replacements will not exhaust the stock.

Example (i). A component is to undergo 450 hours of testing. The mean time to failure of the component is 100 hours and the part will be replaced after 150 hours of service. How many spares should be provided at the test location so that with probability .95 we can complete the specified number of test hours without exhausting the stock?

Example (ii). We have 14 such tests being run simultaneously. How many spares should be provided at one central supply location to service all tests so that with probability .95 we can complete the total $(14)(450) = 6300$ hours of testing without exhausting our stock?

1.) One Equipment

We shall now describe in general terms the problem set out in the example situation (i).

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Let X with or without affixes describe the life length of some component. It is a random variable on the positive real line with distribution, say F , i.e. $F(x)$ is the probability that the component in use will fail on or before time x .

$$(1.1) \quad F(x) = P[X \leq x] \quad x > 0 .$$

If the component of life length X is to be replaced (removed from service) at time t we denote by X^t the truncated life length with distribution G_t (or merely G when no confusion is possible) and we write

$$(1.2) \quad G_t(x) = P[X^t \leq x] = \begin{cases} F(x) & \text{if } x < t \\ 1 & \text{if } x \geq t . \end{cases}$$

Consider a sequence of such truncated life lengths $X_1^t, \dots, X_i^t, \dots$ of spares of one equipment. We assume that the X_i^t are independent and identically distributed by G_t . Let $S_n^t = X_1^t + \dots + X_n^t$. This random variable is the length of time until the n th replacement has occurred whether by in-service breakdown or by scheduled replacement after service of length t . The distribution of S_n^t is the n -fold convolution of the distribution of the X_i^t , i.e.

$$(1.3) \quad G_t^{n*}(x) = P[S_n^t \leq x] = \int_0^x G_t^{(n-1)*}(x-y) d G_t(y) , \quad n = 2, 3, \dots$$

the integral being taken in the Lebesgue-Stieltjes sense, and

$$(1.4) \quad G_t^{1*} = G_t .$$

Let $N^t(x)$ be the total number of replacements up to time x . Now $N^t(x)$ is a non-negative integer valued random variable for each fixed t and x , and it has a distribution, say $H_{t,x}$, (or H when no confusion is possible). Now

$$(1.5) \quad H_{t,x}(n) = P[N^t(x) \leq n] \quad n = 0, 1, \dots$$

One sees that

$$(1.6) \quad P[N^t(x) = n] = P[S_n^t \leq x, S_{n+1}^t > x] = P[S_n^t \leq x] - P[S_{n+1}^t \leq x]$$

hence

$$(1.7) \quad P[N^t(x) > n] = G_t^{(n+1)*}(x) .$$

We wish to determine the minimum number of spares k such that for a specified period of time h and a specified probability $\alpha > 0$ we have

$$(1.8) \quad P[N^t(h) > k] \leq \alpha ,$$

where t is the time of the in-service replacement.

2.) The solution and an approximation for one equipment.

$$(2.1) \quad \text{Let } F_0(x) = \begin{cases} 0 & x < 0 \\ \frac{F(x)}{F(t)} & 0 \leq x < t \\ 1 & x \geq t \end{cases}$$

and

$$(2.2) \quad F_0^0(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} .$$

We will show that

$$(2.3) \quad G_t^{n*}(x) = \sum_{j=0}^n \binom{n}{j} [1 - F(t)]^j [F(t)]^{n-j} F_0^{(n-j)*}(x - jt)$$

thus the calculation of k requires both the knowledge of $F(t)$ and of F_0 .

If G is any distribution on the positive real axis, let \hat{G} be defined by

$$(2.4) \quad \hat{G}(s) = \int_0^\infty e^{-sx} dG(x) .$$

Hence

$$\begin{aligned}\hat{G}_t(s) &= F(t) \hat{F}_0(s) + [1 - F(t)]e^{-st} = \widehat{G_t^{n*}}(s) = [\hat{G}_t(s)]^n = \\ &= \sum_{j=0}^n \binom{n}{j} [1 - F(t)]^j [F(t)]^{n-j} \widehat{F^{(n-j)*}}(s) e^{-jst}\end{aligned}$$

from which the result is clear.

Now (2.3) gives an exact formula needed for the solution of (1.8) using (1.7). This is about all that can be said in general. However the central limit theorem can be used in certain cases. Let

$$(2.5) \quad \mu_t = \int_0^\infty x dG_t(x), \quad \sigma_t^2 = \int_0^\infty (x - \mu_t)^2 dG_t(x)$$

then we know that

$$(2.6) \quad \lim_{n \rightarrow \infty} G_t^{n*}(\sqrt{n}\sigma_t x + n\mu_t) = \Phi(x)$$

where $\Phi(u) = \int_{-\infty}^u \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ is the normal zero-one distribution. Thus

$$(2.7) \quad G_t^{n*}(h) \approx \Phi\left(\frac{h - n\mu_t}{\sqrt{n}\sigma_t}\right)$$

Hence if one knows μ_t and σ_t one can find the least n such that

$$(2.8) \quad h - n\mu_t \leq \sqrt{n}\xi_\alpha \sigma_t$$

where $\xi_\alpha \ni \Phi(\xi_\alpha) = \alpha$ and this is the solution of a quadratic equation. If the solution of (2.8) is n_0 then $k = n_0 - 1$.

3. An Example of type (i).

Let us assume the failure rate of the component is constant in the interval of service $(0, t)$. Hence

$$G_t(x) = \begin{cases} 1 - e^{-\lambda x} & x < t \\ 1 & x \geq t \end{cases}$$

where λ is the failure rate. Then we calculate from (2.5)

$$(3.1) \quad \mu_t = \frac{1 - e^{-\lambda t}}{\lambda} = t - \frac{\lambda t^2}{2} + \frac{\lambda^2 t^3}{3!} + \dots$$

$$(3.2) \quad \begin{aligned} \sigma_t^2 &= (1 - 2\lambda t e^{-\lambda t} - e^{-2\lambda t}) / \lambda^2 \\ &= 3\lambda t^3 - \frac{5}{3}\lambda^2 t^4 + \dots \end{aligned}$$

a) Now take $\mu_t = 100$, $t = 150$, $h = 450$ then $\lambda t = 2/3$ and we find to a linear approximation

$$\lambda = \frac{2}{450} \quad \sigma = 52.2$$

If we set $\alpha = .05$ we find from tables of the normal distribution that $\xi_\alpha = -1.645$ and from (2.8) we wish the least n such that

$$450 - 100n \leq \sqrt{n} (85.8)$$

and we find $n = 7$ satisfies the inequality while $n = 6$ fails. Thus $k = 6$ is the number of spares needed.

b) Now take $\lambda = .01$, $t = 150$, $h = 450$ then from (3.1), (3.2) we find $\mu_t = 77.7$, $\sigma_t = 53.9$ and setting $\alpha = .05$ we find $k = 9$.

Note that these answers are obtained by using the normal approximation and the assumption of constant failure rate during use, not knowing whether or not this is true.

4.) Several tests of the same equipment.

Suppose now that at one location there are several similar, say m , equipments each to be tested a length of time h . Let $N_j(x)$ be the number of replacements of the j th equipment at time x (here we delete the superscript t). Thus at time x the total number of replacements used will be

$$(4.0.1) \quad T_m(x) = N_1(x) + \dots + N_m(x)$$

The problem again is to find the minimum number k of spares needed at this one location such that if each equipment undergoes a test of length h with scheduled replacement at time t of the components.

We have k being the least integer n such that

$$P[T_m(h) > k] \leq \beta$$

where β is a probability specified in advance.

Again we define

$$(4.1) \quad P[T_m(h) \leq n] = H_{t,h}^{m*}(n) \quad n = 0, 1, \dots$$

where $H_{t,h}$ was defined in (1.5). Now in the convolution formula we have

$$(4.2) \quad H_{t,x}^{m*}(n) = \sum_{j=0}^n H_{t,x}^{(m-1)*}(n-j) [H_{t,x}(j) - H_{t,x}(j-1)]$$

$$(4.3) \quad = \sum_{j=0}^n H_{t,x}^{(m-1)*}(n-j) [G_t^j(x) - G_t^{(j+1)*}(x)]$$

which is a recursion formula which can be used for machine computation.

In an actual problem m, t, x are given and one finds the least n via computation from (4.2) such that $H_{t,x}^{m*}(n) \leq \beta$.

Let us write the expected number of failures at time x as

$$(4.3.1) \quad U_t(x) = \mathcal{E} N^t(x) = \sum_{n=0}^{\infty} n P[N^t(x) = n]$$

omitting the arguments of the function

$$U = \sum_{n=0}^{\infty} n [G^{n*} - G^{(n+1)*}] = \sum_{n=1}^{\infty} G^{n*}.$$

Thus U satisfies the integral equation

$$(4.3.2) \quad U = G + U * G.$$

Let $V_t(x) = \mathcal{E} [N^t(x)]^2 = \sum_{n=0}^{\infty} n^2 P[N^t(x) = n]$ then

$$V = 2 \sum_{n=1}^{\infty} n G^{n*} - U$$

and V satisfies the integral equation

$$(4.3.3) \quad V = U + V * G$$

Hence we have from (4.3.2) and (4.3.3)

$$\hat{U} = \frac{\hat{G}}{1 - \hat{G}} \quad \hat{V} = \frac{\hat{U}}{1 - \hat{G}} = \frac{\hat{G}}{(1 - \hat{G})^2}.$$

Now

$$\hat{G}_t(s) = 1 - \mu_t s + (\sigma_t^2 + \mu_t^2) \frac{s^2}{2} + O(s^3).$$

Thus we have from known relationships concerning the corresponding behavior

of the Laplace transform near zero and the generating function near infinity

$$\lim_{x \rightarrow \infty} \frac{U_t(x)}{x} = \frac{1}{\mu_t}, \quad \lim_{x \rightarrow \infty} \frac{V_t(x) - U_t^2(x)}{x} = \frac{\sigma_t^2}{\mu_t^3}$$

Let $d_t^2(x) = V_t(x) - U_t^2(x)$ and for x large we have

$$(4.4) \quad U_t(x) \approx \frac{x}{\mu_t} \quad d_t^2(x) \approx \frac{\sigma_t^2}{3\mu_t} x.$$

Let U_0 be any guess as to what U is. Now define

$$(4.4.1) \quad U_{n+1} = G + U_n * G$$

then it follows that $U_n \rightarrow U$.

Let V_0 be any guess as to what V is. Now define

$$(4.4.2) \quad V_{n+1} = G + V_n * [2G - G^{2*}]$$

then it follows that $V_n \rightarrow V$.

These iteration procedures can be used to find $V(h)$ and $d(h)$ instead of the crude approximations (4.4). Unfortunately these procedures require machine programming.

Now, of course, if one central location is to provide spares for all locations and there are m_i equipments being listed at the i th location $i = 1, \dots, r$ then at time x the total number of replacements used will be

$$W(x) = T_{m_1}(x) + \dots + T_{m_r}(x).$$

This random variable has the distribution

$$H_{t,n}^{v*} \quad \text{where} \quad v = \sum_{i=1}^r m_i.$$

which is exactly the same distribution as given in (4.1), (4.2) except that the degree of convolution is higher.

Notice from (4.0.1) we may regard

$$W(x) = N_1(x) + \dots + N_v(x)$$

with all the summands independent and identically distributed.

We wish to find least k such that

$$P[W(x) > k] \leq \gamma$$

for $\gamma > 0$ previously given. This is equivalent with finding the least k such that

$$H_{t,n}^{v*}(k-1) \geq 1 - \gamma.$$

Thus if v is sufficiently large we may use the same reasoning § 2 to set,

$$(4.5) \quad H_{t,n}^{v*}(n) \cong \Phi \left[\frac{n + .5 - vU_t(h)}{\sqrt{v} d_t(h)} \right]$$

and we proceed as before except that the variable is changed and increased by .5 for the discrete approximation.

If we are content to use the approximation (4.4), we seek the least integer k such that

$$(4.6) \quad k \geq v \frac{h}{\mu_t} + \sqrt{v} \frac{\sigma_t}{\mu_t} \sqrt{\frac{h}{\mu_t}} \xi_{1-\gamma} + \frac{1}{2}$$

as the approximate solution to the problem. We should realize that (4.6) will probably not be as nearly good as that in (2.8) because we have used the approximation of (4.4) as well as the approximation of the convolution of a discrete random variable by the normal. However (4.6) prescribes quick and easy approximations which could be used as a first guess in beginning the iteration in equation (4.2).

5.) An example of type (ii)

Let us make for comparisons sake the same assumption of constant failure rate that was made in § 3 and further make the assumption that the parameter values are the same. Take $v = 14$, $\mu_t = 100$, $h = 450$, $\sigma_t = 52.2$ then from (4.4) we calculate

$$U_t(h) = 4.5 \quad d_t(h) = 1.11 .$$

By setting $\gamma = .95$ then $\xi_{1-\gamma} = 1.645$ and from equation (4.6) we have

$$n \geq 35.2$$

hence $k = 36$.

Using these approximations we find one can supply 14 test locations jointly with the same stock of spares that could be used to supply 4 individually. This amazing result probably indicates nothing more than the crudeness of the approximation (4.4). We should of course expect to use less jointly than the mere product of the number of locations times the number of spares at each site however the result seems a little too good. Its validity must be checked by use of the exact formulae (4.3) provided the exact information is available.

6.) Comments

To solve the problem exactly via formulae (2.3) or (4.3) we need to know the distribution F on the interval $(0, t)$ moreover the use of these formulae would in general require machine programming for their solution. To make the normal approximation of (2.7) we need to know considerably less only μ_t and σ_t .

If F is known on $(0, t)$ one can use iteration procedures to find U and V as given in (4.4.1) and (4.4.2) and then use the normal approximation.

There is another situation which one may obtain and that is a set of observations of the random variable X_n^t from which we must make statistical inference about the number of spares required. That is one can only estimate the distribution F in $(0, T)$. This is another topic in which very little is known.

All of the relevant material on probability that is used in the preceding discussion can be found in the following texts:

An Introduction to Stochastic Processes, M. S. Bartlett, Cambridge University Press, 1955, pp.20-24.

Stochastic Processes, Lajos Takács, John Wiley & Sons, 1960, Chapter 3.